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Polynomial invariants of graphs with state models

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Abstract

We reformulate a polynomial invariant of graphs defined by Negami, using the notion of state models, and discuss another polynomial invariant, as a natural extension of Negami's polynomial, which can distinguish many graphs more finely than the original.

1. Introduction

Negami [3] defined a polynomial $f(G) = f(G; t, x, y)$ with three variables t, x, y for each graph G, called the *Negami polynomial* here, as one that satisfies the following conditions:
(i) $f(\overline{K_n}) = t^n$,
(ii) $f(G) = xf(G/e) + yf(G - e)$ $(e \in E(G))$.
Here $\overline{K_n}$ is the complement of the complete graph K_n with n vertices, i.e. a graph consisting of only n isolated points with no edge. The two graphs G/e and $G - e$ are obtained from G by *contraction* and *deletion* of an edge e. Contraction of e is to identify the ends of e to one vertex after deleting e. We do not remove multiple edges yielded by contraction of edges.

This polynomial $f(G)$ expands into the form $\sum a_{ij} t^j x^i y^{|E(G)| - i}$ and each coefficient a_{ij} coincides with the number of spanning subgraphs of G with precisely i edges and j components. Translating this relationship between degrees and coefficients, we can recognize many things on the structure of G; for example, the number of vertices, edges, components, self-loops, spanning trees and complete subgraphs, and the edge connectivity, the chromatic number, being eulerian or not and so on.

Our purpose in this paper is to reconstruct $f(G)$ from a completely different point of view. Negami has focused on the edges of graphs to calculate $f(G)$ in [3]. On the other

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hand, we pay attention to the "states" of vertices, mimicking Kauffman's idea of state models. He introduced state models for analyzing many polynomial invariants of knots and links, related to statistical mechanics. (See [2] for example. We can find a survey on similar arguments in [1].) Roughly speaking, an invariant with a state model is defined as the summation of weights of an object in question taken over all the states of it.

We shall show that $f(G)$ can be obtained as an invariant with a suitable state model and define a new polynomial $\tilde{f}(G) = \tilde{f}(G;t,x,z,y)$ with four variables t,x,z,y as a natural extension of $f(G)$ with the same idea. For example, the Negami polynomial and its extension of the complete graph K_3 with three vertices are as follows:

$$f(K_3;t,x,y) = tx^3 + 3tx^2y + 3t^2xy^2 + t^3y^3,$$

$$\tilde{f}(K_3;t,x,z,y) = (x+y)^3 + (t-1)(z+y)^3 \\ + 3\{(t-1)(x+y)y^2 + (t-1)^2(z+y)y^2\} + t(t-1)(t-2)y^3.$$

Substitute $z = x$ in the above $\tilde{f}(K_3)$ to get a polynomial with three variables t,x,y. Then we get the same polynomial as the original Negami polynomial $f(K_3)$. In fact, we shall show that $\tilde{f}(G;t,x,x,y) = f(G;t,x,y)$ for any graph G.

This equality implies that if $\tilde{f}(G) = \tilde{f}(G')$, then $f(G) = f(G')$. So if we can show an example of a pair of graphs G and G' with $f(G) = f(G')$ but $\tilde{f}(G) \neq \tilde{f}(G')$, then $\tilde{f}(G)$ can be said to be stronger than $f(G)$. Negami has already discussed in [3] what kind of deformations of graphs preserve $f(G)$ and showed that $f(G)$ is a 2-isomorphism invariant rather than an isomorphism invariant, roughly speaking. For example, split a graph G into two graphs at two vertices and join them again by different identification. Then we get another graph with the same $f(G)$. We shall show that this deformation does not preserve $\tilde{f}(G)$ in general.

Let U be a subset of $V(G)$ and $\bar{U} = V(G) - U$ its complement. Let $[U,\bar{U}]$ be the set of edges uv with $u \in U$ and $v \in \bar{U}$, and let $e(U)$ denote the number of edges both of whose ends belong to U. Our new polynomial $\tilde{f}(G)$ is defined formally as

$$\tilde{f}(G;t,x,z,y) = \sum_{U \subset V(G)} f(G-U;t-1,z,y)y^{|[U,\bar{U}]|}(x+y)^{e(U)}.$$

This is so complicated that nobody can recognize easily the above-mentioned facts on $\tilde{f}(G)$, but everything works naturally through the state models. For example, we can understand via our state model why $f(\overline{K_n})$ should be defined as t^n and what the variable t is.

The Tutte polynomial $T(G)$ is a famous invariant of graphs, related to matroid theory. In fact, $T(G)$ can be derived from $f(G)$ by suitable substitution of variables. Thus, $\tilde{f}(G)$ can be said to be stronger than $T(G)$, too. We shall discuss the relationship among these three polynomials $f(G)$, $\tilde{f}(G)$ and $T(G)$ in more detail in Section 3.

2. Polynomials with state models

Let S be a fixed finite set and let G be any graph. We call each element $s \in S$ a *state* and any map $\sigma: V(G) \to S$ a *state function*, which assigns a state to each vertex of G. Let R be a commutative ring with identity 1 and $w: S \times S \to R$ a *symmetric map*, i.e.

$$w(a,b) = w(b,a) \in R \quad (a, b \in S).$$

We define a ring element $Z_w(G) \in R$ as follows:

$$Z_w(G) = \sum_{\sigma} \prod_{uv \in E(G)} w(\sigma(u), \sigma(v)),$$

where $\sigma: V(G) \to S$ runs over all the state functions.

The *weight* $w(\sigma(u), \sigma(v))$ of each edge with ends u and $v \in V(G)$ appears exactly once in the product

$$W_{w\sigma}(G) = \prod_{uv \in E(G)} w(\sigma(u), \sigma(v)).$$

If there exist n parallel edges with the same ends u and v, then $W_{w\sigma}(G)$ is divisible by $w(\sigma(u), \sigma(v))^n$. If $E(G) = \emptyset$, then $W_{w\sigma}(G)$ is the empty product and so is equal to the identity $1 \in R$ for each state function $\sigma: V(G) \to S$. Thus, we have

$$Z_w(\overline{K_n}) = \sum_{\sigma} 1 = |S|^n$$

for any weight function $w: S \times S \to R$.

It is clear that $Z_w(G)$ is an isomorphism invariant of graphs, i.e. if two graphs G and G' are isomorphic, then $Z_w(G) = Z_w(G')$. Our weight function $w: S \times S \to R$ is supposed to be symmetric. If we use an anti-symmetric weight function, $Z_w(G)$ will become an invariant for directed graphs. For example, define a weight function $D: S \times S \to \boldsymbol{Z}[x, y_+, y_-]$ with $S = \{1, 2, \ldots, n\}$ by

$$D(a,b) = \begin{cases} x & (a = b), \\ y_+ & (a < b), \\ y_- & (a > b). \end{cases}$$

Then $Z_D(G) = \sum_\sigma \prod_{uv \in E(G)} D(\sigma(u), \sigma(v))$ will be a polynomial with three variables x, y_+ and y_-. It should be noticed that $uv \in E(G)$ in the product runs through all the directed edges of G. However, we shall not discuss such an invariant in this paper.

Now define a weight function $\chi: S \times S \to \boldsymbol{Z}$ by

$$\chi(a,b) = \begin{cases} 0 & (a = b), \\ 1 & (a \neq b). \end{cases}$$

Then we have an integer $Z_\chi(G)$ for each graph G.

$$Z_\chi(G) = \sum_\sigma \prod_{uv \in E(G)} \chi(\sigma(u), \sigma(v)) = \sum_\sigma W_{\chi\sigma}(G).$$

If at least one edge has two ends with the same state for a state function σ, then $W_{\chi\sigma}(G) = 0$. Thus, the product $W_{\chi\sigma}(G)$ does not vanish and is equal to 1 when and only when σ assigns states to vertices so that two vertices have two different states if they are adjacent. This implies that such a σ is a vertex coloring with color set S and that $Z_\chi(G)$ is equal to the total number of vertex colorings of G with $|S|$ colors. Thus, $Z_\chi(G)$ coincides with the chromatic polynomial $P(G;t)$ of G with $t = |S|$.

For the chromatic polynomial $P(G;t)$, we have the following recursive formula.

$$P(G;t) = P(G-e;t) - P(G/e;t).$$

This is very similar to the definition of $f(G)$. Actually, substitution of $x = -1$ and $y = 1$ reduces $f(G;t,x,y)$ into $P(G;t)$, so $f(G)$ can be regarded as an extension of the chromatic polynomial. Then we would like to redefine the Negami polynomial $f(G)$, extending the definition of $Z_\chi(G)$ as the following theorem shows.

Theorem 1. *Let S be a finite set of states and $\eta : S \times S \to \mathbf{Z}[x,y]$ a weight function with polynomial values defined by*

$$\eta(a,b) = \begin{cases} x + y & (a = b), \\ y & (a \neq b). \end{cases}$$

Then $Z_\eta(G) = f(G;|S|,x,y)$.

Proof. Let $e = uv \in E(G)$ be any edge of G. By the definition, we have

$$\begin{aligned} Z_\eta(G) &= \sum_\sigma W_{\eta\sigma}(G) \\ &= \sum_{\sigma(u)=\sigma(v)} W_{\eta\sigma}(G) + \sum_{\sigma(u)\neq\sigma(v)} W_{\eta\sigma}(G), \\ Z_\eta(G/e) &= \sum_{\sigma(u)=\sigma(v)} W_{\eta\sigma}(G)/\eta(\sigma(u),\sigma(v)) \\ &= \frac{1}{x+y} \sum_{\sigma(u)=\sigma(v)} W_{\eta\sigma}(G), \\ Z_\eta(G-e) &= \sum_\sigma W_{\eta\sigma}(G)/\eta(\sigma(u),\sigma(v)) \\ &= \frac{1}{x+y} \sum_{\sigma(u)=\sigma(v)} W_{\eta\sigma}(G) + \frac{1}{y} \sum_{\sigma(u)\neq\sigma(v)} W_{\eta\sigma}(G). \end{aligned}$$

From these, we get the following formulas:

(i) $Z_\eta(\overline{K_n}) = |S|^n$,

(ii) $Z_\eta(G) = xZ_\eta(G/e) + yZ_\eta(G-e)$ $(e \in E(G))$.

These are the same conditions as in the definition of $f(G)$ if $t = |S|$. Therefore, $Z_\eta(G)$ coincides with the two-variable polynomial $f(G; |S|, x, y)$. $\square$

Although, from its definition, $Z_\eta(G)$ does not look like a polynomial in $|S|$, the above theorem means that $Z_\eta(G)$ can be expressed as a polynomial in $|S|$, by substituting $t = |S|$ in $f(G; t, x, y)$.

Now we shall define a new polynomial $\tilde{f}(G, t, x, z, y)$ with another state model, by modifying the weight function $w: S \times S \to \boldsymbol{Z}[x, y]$. Let $S = \{1, 2, 3, \ldots, t\}$ be a finite set of precisely t states and define a weight function $\tilde{\eta}: S \times S \to \boldsymbol{Z}[x, z, y]$ by

$$\tilde{\eta}(a,b) = \begin{cases} x + y & (a = b = 1), \\ z + y & (a = b \neq 1), \\ y & (a \neq b). \end{cases}$$

According to our general theory, we define a polynomial $Z_{\tilde{\eta}}(G) = Z_{\tilde{\eta}}(G; x, z, y)$ with three variables x, z, y as follows:

$$Z_{\tilde{\eta}}(G) = \sum_{\sigma} \prod_{uv \in E(G)} \tilde{\eta}(\sigma(u), \sigma(v)).$$

Lemma 2. *$Z_{\tilde{\eta}}(G)$ can be expressed as a polynomial with variable t.*

Proof. Any state function $\sigma: V(G) \to \{1, 2, 3, \ldots, t\}$ is uniquely determined by specifying the subset $U = \sigma^{-1}(1) \subset V(G)$ and the map $\sigma' = \sigma|_{V(G)-U}: V(G) - U \to \{2, 3, \ldots, t\}$. For this state function σ, the weight function $\tilde{\eta}$ assigns y to each edge uv with ends $u \in U$ and $v \in \bar{U} = V(G) - U$ and $x + y$ to each edge with both ends in U. Thus, we have the following expansions:

$$\begin{aligned} Z_{\tilde{\eta}}(G) &= \sum_{\sigma} \prod_{uv \in E(G)} \tilde{\eta}(\sigma(u), \sigma(v)) \\ &= \sum_{U \subset V(G)} \sum_{\sigma'} \prod_{uv \in E(G-U)} \tilde{\eta}(\sigma'(u), \sigma'(v)) y^{|[U, \bar{U}]|} (x + y)^{e(U)}. \end{aligned}$$

Comparing $\sum \prod \tilde{\eta}(\sigma'(u), \sigma'(v))$ with the definition of $Z_\eta(G) = f(G; |S|, x, y)$, we have

$$Z_{\tilde{\eta}}(G) = \sum_{U \subset V(G)} f(G - U; t - 1, z, y) y^{|[U, \bar{U}]|} (x + y)^{e(U)}.$$

Since $f(G - U; t - 1, z, y)$ is a polynomial in t, so is $Z_{\tilde{\eta}}(G)$. $\square$

The polynomial expression of $Z_{\tilde{\eta}}(G)$ with four variables t, x, z, y is called here the *extended Negami polynomial* and is denoted by $\tilde{f}(G; t, x, z, y)$. By the previous proof, $\tilde{f}(G; t, x, z, y)$ can be defined with the following expansion:

$$\tilde{f}(G; t, x, z, y) = \sum_{U \subset V(G)} f(G - U; t - 1, z, y) y^{|[U, \bar{U}]|} (x + y)^{e(U)}.$$

Here t is an independent variable and is not a constant equal to the number of states. Notice that $\tilde{f}(G)$ does not satisfy the same simple recursive formula as $f(G)$.

The following equality can be easily shown via our state model, but it will be difficult to derive it from the above formula only.

Theorem 3. $\tilde{f}(G; t, x, x, y) = f(G; t, x, y)$.

Proof. Since the definition of $\tilde{\eta}$ with $z = x$ determines the same weight function as $\eta, \tilde{f}(G; t, x, x, y)$ has to coincide with $f(G)$. □

One of the main results in [3] is a "splitting formula" for $f(G)$, which shows that if a graph G splits into two graphs H and K having only several vertices in common, $f(G)$ splits nicely, depending only on the structures of H and K. From this formula, Negami showed that any 2-isomorphic graphs with the same number of components have the same Negami polynomial. On the other hand, our new polynomial $\tilde{f}(G)$ does not split nicely, i.e. $\tilde{f}(H \cup K)$ depends not only on H and K but also how they attach to each other, as is shown later. We have only the following splitting formula for $\tilde{f}(G)$.

Theorem 4. *Let $K \cup H$ be a disjoint union of two graphs H and K. Then*

$$\tilde{f}(H \cup K) = \tilde{f}(H)\tilde{f}(K).$$

Proof. When K and H are disjoint from each other, a state function $\sigma: V(K \cup H) \to S$ can be determined by giving two restrictions $\sigma_K: V(K) \to S$ and $\sigma_H: V(H) \to S$ independently. Thus, we have

$$\begin{aligned}
Z_{\tilde{\eta}}(K \cup H) &= \sum_{\sigma} W_{\tilde{\eta}\sigma}(K \cup H) \\
&= \sum_{\sigma} W_{\tilde{\eta}\sigma_K}(K) W_{\tilde{\eta}\sigma_H}(H) \\
&= \sum_{\sigma_K} \sum_{\sigma_H} W_{\tilde{\eta}\sigma_K}(K) W_{\tilde{\eta}\sigma_H}(H) \\
&= \left(\sum_{\sigma_K} W_{\tilde{\eta}\sigma_K}(K) \right) \left(\sum_{\sigma_H} W_{\tilde{\eta}\sigma_H}(H) \right) \\
&= Z_{\tilde{\eta}}(K) Z_{\tilde{\eta}}(H).
\end{aligned}$$

This implies the theorem. □

3. Discriminating graphs with polynomials

Here we shall discuss the discriminating power of our polynomials $f(G)$ and $\tilde{f}(G)$. Since $\tilde{f}(G)$ is an extension of $f(G)$, we expect that $\tilde{f}(G)$ can distinguish many graphs which $f(G)$ cannot do. To construct and discuss examples for such graphs, we shall define several deformations as follows.

Let H and K be two disjoint graphs and let $u_1, \ldots, u_n \in V(H)$ and $v_1, \ldots, v_n \in V(K)$. Identify $u_1 = v_1, \ldots, u_n = v_n$ in the disjoint union $H \cup K$ of H and K. We denote the resulting graph by $H \cup K/(u_1 = v_1, \ldots, u_n = v_n)$.

(i) Addition or deletion of isolated points: $H \leftrightarrow H \cup K_1$.
(ii) Split or join at one vertex: $H \cup K/(u_1 = v_1) \leftrightarrow H \cup K$.
(ii)′ Arrangement of blocks: $H \cup K/(u_1 = v_1) \leftrightarrow H \cup K/(u_2 = v_2)$.
(iii) Turning around two vertices:

$$H \cup K/(u_1 = v_1, u_2 = v_2) \leftrightarrow H \cup K/(u_1 = v_2, u_2 = v_1).$$

Whitney [8] showed that two graphs G and G' can be transformed into each other by (i)–(iii) if and only if there is a bijection $\phi: E(G) \to E(G')$ which induces a bijection between the collections of cycles in G and in G' and two such graphs are said to be *2-isomorphic* to each other. In fact, (i) and (ii) do not destroy any cycle in a graph even if they change the number of components. On the other hand, (iii) preserves the set of edges forming a cycle but breaks the order of edges lying on the cycle. So a cycle should be regarded here as a set of edges. Since neither (i) nor (iii) is applicable to any 3-connected graph, two 3-connected graphs are 2-isomorphic if and only if they are isomorphic, which is proved in [7].

In the present terminology, the composite structure $(E(G), \mathscr{C}(G))$ with the edge set $E(G)$ and the collection $\mathscr{C}(G)$ of cycles in G is called the *cycle matroid* of G. (See [6] for general theory of matroids.) Thus, two graphs are 2-isomorphic if and only if their cycle matroids are isomorphic. As a 2-isomorphism invariant or a matroid invariant, we know the *Tutte polynomial* $T(G) = T(G; x, y)$ with two variables x, y, i.e. if G and G' are 2-isomorphic, then $T(G) = T(G')$. (The Tutte polynomial $T(G)$ is called the *dichromatic polynomial* in [5].)

Let $\omega(G)$ denote the number of components of a graph G. Since the minimum degree of $f(G)$ in t is equal to $\omega(G)$, $f(G)$ is not a 2-isomorphism invariant, precisely speaking. However, (ii)′ and (iii) preserve $f(G)$, as is shown in [3]. Thus, if G and G' are 2-isomorphic graphs with the same number of components, then $f(G) = f(G')$.

In fact, the 2-isomorphism invariant $T(G)$ can be obtained from $f(G)$ by omitting information on the numbers of vertices and of components, as follows:

$$T(G; x, y) = f(G; (x-1)(y-1), 1, y-1)(y-1)^{-|V(G)|}(x-1)^{-\omega(G)}.$$

Conversely, Oxley [4] showed that $f(G)$ also can be obtained from $T(G)$.

$$f(G; t, x, y) = T\left(G; 1 + \frac{ty}{x}, 1 + \frac{x}{y}\right)\left(\frac{ty}{x}\right)^{\omega(G)}\left(\frac{x}{y}\right)^{|V(G)|} y^{|E(G)|}.$$

So they can be said to be nearly equivalent although there exist graphs G and G' such that $T(G) = T(G')$ and $f(G) \neq f(G')$.

Now compare $f(G)$ and $\tilde{f}(G)$. The deformations (i) and (ii) do not preserve $\tilde{f}(G)$ as well as $f(G)$ since $\tilde{f}(G; t, x, x, y) = f(G; t, x, y)$. Since (ii)′ can be realized as a special type of (iii) with $v' \in V(K)$ isolated in K, it suffices to discuss whether or not (iii) preserves $\tilde{f}(G)$. So we shall consider the minimum degree $\delta(G)$ of G as what (iii) does not preserve.

Theorem 5. *If a graph G has no loops and no isolated points, then the minimum degree $\delta(G)$ is determined by $\tilde{f}(G)$.*

Proof. Let X and Z be two independent variables and substitute $x = X - y$, $z = Z - y$ in $Z_{\tilde{\eta}}(G) = Z_{\tilde{\eta}}(G; X, Z, y)$ can be regarded as the polynomial defined with a weight function $\tilde{\eta}: S \times S \to \boldsymbol{Z}[X, Z, y]$

$$\tilde{\eta}(a,b) = \begin{cases} X & (a = b = 1), \\ Z & (a = b \neq 1), \\ y & (a \neq b). \end{cases}$$

Thus, $Z_{\tilde{\eta}}(G)$ expands into the form

$$Z_{\tilde{\eta}}(G; X, Z, y) = \sum_{\sigma} X^{l_\sigma} Z^{m_\sigma} y^{n_\sigma},$$

where $l_\sigma + m_\sigma + n_\sigma = |E(G)|$. Consider a state function σ such that $m_\sigma = 0$, $n_\sigma > 0$ and that n_σ is minimal among such state functions.

Since G is not isomorphic to $\overline{K_{|V(G)|}}$ and $n_\sigma > 0$, there are at least two vertices to which σ assigns two different states. Choose v as one of those with $\sigma(v) \neq 1$, say $\sigma(v) = 2$. Then all the edges incident to v have weight y since $m_\sigma = 0$. By the minimality of n_σ, there is no other edge with weight y; otherwise, we could eliminate such edges with another state function σ' which assigns 1 to all vertices except v, and $n_{\sigma'} < n_\sigma$. Thus, we have $\deg v = n_\sigma$ and this value has to be equal to the minimum degree $\delta(G)$. □

Let H and K be two disjoint simple graphs and $u_1, u_2 \in V(H)$ and $v_1, v_2 \in V(K)$. Let $G_1 = H \cup K/(u_1 = v_1, u_2 = v_2)$ and $G_2 = H \cup K/(u_1 = v_2,\ u_2 = v_1)$. Suppose that

$$\delta(G_1) = \deg_H u_1 + \deg_K v_1,$$

$$\deg_H u_1, \deg_K v_1 < \deg_H u_2, \deg_K v_2.$$

Then $\delta(G_1) < \delta(G_2)$ and hence $\tilde{f}(G_1) \neq \tilde{f}(G_2)$ by the above theorem while $f(G_1) = f(G_2)$ since G_1 and G_2 are 2-isomorphic. Therefore, we can say that the extended Negami polynomial $\tilde{f}(G)$ is actually stronger than the original $f(G)$.

Finally we shall define an infinite sequence of polynomials $f_1(G), f_2(G), f_3(G), \ldots$ such that $f_{n+1}(G)$ is an extension of $f_n(G)$ for each $n = 1, 2, 3, \ldots$. To do so, we define a weight function $\eta_n : S \times S \to \boldsymbol{Z}[x_1, \ldots, x_n, y]$ with $S = \{1, 2, \ldots, t\}$ by

$$\eta_n(a,b) = \begin{cases} x_i + y & (a = b = i < n), \\ x_n + y & (a = b \geqslant n), \\ y & (a \neq b). \end{cases}$$

Then we have a polynomial $Z_{\eta_n}(G)$ with $n+1$ variables $x_1, \ldots, x_n, y$ by our general theory. The same argument as on $\tilde{f}(G)$ concludes that the coefficient of each term in $Z_{\eta_n}(G)$ can be expressed as a polynomial in t. So we get a new polynomial $f_n(G; t, x_1, \ldots, x_n, y)$ as such a polynomial expression of $Z_{\eta_n}(G)$ with $n+2$ variables $t, x_1, \ldots, x_n, y$. Clearly, we have the following by definition:

$$f_1(G; t, x_1, y) = f(G; t, x_1, y),$$

$$f_2(G; t, x_1, x_2, y) = \tilde{f}(G; t, x_1, x_2, y),$$

$$f_{n+1}(G; t, x_1, \ldots, x_n, x_n, y) = f_n(G; t, x_1, \ldots, x_n, y).$$

Thus, if $f_{n+1}(G) = f_{n+1}(G')$, then $f_n(G) = f_n(G')$. Is $f_{n+1}(G)$ actually stronger than $f_n(G)$? That is, do there exist two graphs G and G' such that $f_n(G) = f_n(G')$ and $f_{n+1}(G) \neq f_{n+1}(G')$?

References

[1] P. de la Harpe and V.F.R. Jones, Graph invariants related to statistical mechanical models: examples and problems, J. Combin. Theory Ser. B. 57 (1993) 207–227.
[2] L.H. Kauffman, Knots and Physics (World Scientific Publishing, Singapore, 1991).
[3] S. Negami, Polynomial invariants of graphs, Trans. Amer. Math. Soc. 299 (1987) 601–622.
[4] J.G. Oxley, A note on Negami's polynomial invariants for graphs, Discrete Math. 76 (1989) 279–281.
[5] W.T. Tutte, Graph Theory, Encyclopedia of Mathematics 21 (Addison-Wesley, Reading, MA, 1984).
[6] D.J.A. Welsh, Matroid Theory (Academic Press, London, 1976).
[7] H. Whitney, Non-separable and planar graphs, Trans. Amer. Math. Soc. 34 (1932) 339–362.
[8] H. Whitney, 2-isomorphic graphs, Amer. J. Math. 55 (1933) 245–254.